TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 244, October 1978

GENERALIZED CONJUGATE FUNCTION THEOREMS FOR SOLUTIONS OF FIRST ORDER ELLIPTIC SYSTEMS ON THE PLANE

BY

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ABSTRACT. Our essential aim is to generalize Privoloff's theorem, Schwarz reflection principle, Kolmogorov's theorem and the theorem of M. Riesz for conjugate functions to the solutions of differential equations in the $z = x + iy$ plane of the following elliptic type:

$$\text{(M)} \qquad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = au + bv + f, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = cu + dv + g.$$

THEOREM 1. *Let the coefficients of* (M) *be Hölder continuous on* $|z| \leq 1$. *Let* (u, v) *be a solution of* (M) *in* $|z| < 1$. *If* u *is continuous on* $|z| \leq 1$ *and Hölder continuous with index* α *on* $|z| = 1$, *then* (u, v) *is Hölder continuous with index* α *on* $|z| \leq 1$.

THEOREM 2. *Let the coefficients of* (M) *be continuous on* $|z| \leq 1$ *and satisfy the condition*

$$\text{(N)} \qquad \int_0^y b(x,t)\,dt + \int_0^x d(t,y)\,dt = \int_0^y b(0,t)\,dt + \int_0^x d(t,0)\,dt$$

for $|z| \leq 1$. *And let* $\|f\|_p = \sup_{0<r<1}\{(1/2\pi)\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta\}^{1/p}$. *Then to each* p, $0 < p < \infty$, *there correspond two constants* A_p *and* B_p *such that*

$$\|v\|_p \leqslant A_p\|u\|_p + B_p, \quad 1 < p < \infty,$$
$$\|v\|_p \leqslant A_p\|u\|_1 + B_p, \quad 0 < p < 1,$$

hold for every solution (u, v) *of* (M) *in* $|z| < 1$ *with* $v(0) = 0$. *If* $f \equiv g \equiv 0$, *the theorem holds for* $B_p = 0$. *Furthermore, if* b *and* d *do not satisfy the condition* (N) *in* $|z| \leq 1$, *then we can relax the condition* $v(0) = 0$, *and still have the above inequalities.*

THEOREM 3. *Let the coefficients of* (M) *be analytic for* x, y *in* $|z| < 1$. *Let* (u, v) *be a solution of* (M) *in* $\{|z| < 1\} \cap \{y > 0\}$. *If* u *is continuous in* $\{|z| < 1\} \cap \{y \geq 0\}$ *and analytic on* $\{-1 < x < 1\}$, *then* (u, v) *can be continued analytically across the boundary* $\{-1 < x < 1\}$. *Furthermore, if the coefficients and* u *satisfy some further boundary conditions, then* (u, v) *can be continued analytically into the whole of* $\{|z| < 1\}$.

Received by the editors February 28, 1972 and, in revised form, February 24, 1976.

AMS (MOS) subject classifications (1970). Primary 35J44, 35B99, 35J55, 35J67, 30A93.

Key words and phrases. First order elliptic equations, pseudoanalytic functions, Cauchy-Riemann equations, Privoloff's theorem, Schwarz reflection principle, Kolmogorov's theorem, the theorem of M. Riesz for conjugate functions.

1. **Introduction.** The intimate relation between the real part and the imaginary part of a holomorphic function has been widely investigated, both for its own interest and for its importance in applications. Generally speaking, the real part and the imaginary part of a holomorphic function behave alike, although there are some rather surprising exceptions. Attempts to demonstrate similarities between the two led to many important results such as Privoloff's theorem, Schwarz reflection principle, the theorem by M. Riesz for conjugate functions, and Kolmogorov's theorem for conjugate functions.

The main purpose of this paper is to extend the above-mentioned theorems to the solutions of the elliptic equations in the $z = x + iy$ plane of the following type:

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= au + bv + f, \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= cu + dv + g. \end{aligned} \tag{1.1}$$

The above problems are interesting since solutions of equation (1.1) have many properties similar to those of holomorphic functions. Some of those properties have been studied systematically by Bers [3] and Vekua [11], [12].

The Privoloff theorem states:

THEOREM 1.1 (PRIVOLOFF [4]). *If $f(z)$ for $|z| < 1$ is holomorphic, and $\operatorname{Re} f(z)$ is continuous for $|z| \leqslant 1$ and satisfies a Hölder condition with exponent $\delta < 1$ and constant K on $|z| = 1$, then $f(z)$ satisfies on $|z| \leqslant 1$ a Hölder condition with exponent δ and constant CK, where C depends only on δ.*

This theorem has been generalized to solutions of equation (1.1) by Vekua [12] under Riemann-Hilbert boundary conditions. Agmon, Douglis and Nirenberg [1], [2] have generalized it to the solutions of the general elliptic equations under ' complementing boundary conditions.'

The Schwarz reflection principle can be stated as follows:

THEOREM 1.2. *Let Ω^+ be the part in the upper half-plane of a symmetric region Ω, and let σ be the part of the real axis in Ω. Suppose that $f(z)$ is holomorphic in Ω^+, $\operatorname{Re} f(z)$ is continuous in $\Omega^+ \cup \sigma$, and zero on σ; then $f(z)$ has an analytic extension to Ω.*

The above theorem has been generalized by many authors to solutions of various types of elliptic equations with analytic coefficients in the plane (Lewy [7], Garabedian [6]). Yu [13], [14] has obtained reflection principles for (1.1) with analytic coefficients under linear or nonlinear analytic Riemann-Hilbert type boundary conditions, and the references can be found there.

The main thing we want to emphasize about the Privoloff theorem and the

Schwarz reflection principle is that there is nothing at all that needs to be assumed about the boundary behavior of the imaginary part. However, in the generalizations of Vekua [12] and Yu [13] to the equations (1.1) under Riemann-Hilbert type boundary conditions, they both assumed $u(z)$ and $v(z)$ are continuous up to the boundary. Thus, it is important to study the case that $u(z)$ satisfies the boundary conditions corresponding to $\operatorname{Re} f(z)$ in the Privoloff theorem and the Schwarz reflection principle, and assume nothing about the $v(z)$ at the boundary. These will be investigated in §§4 and 5.

Let $U = \{|z| < 1\}$. For any f defined on U, we put

$$\|f\|_p = \lim_{r\to 1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta\right\}^{1/p}, \qquad 0 < p < 1. \tag{1.2}$$

The theorem of M. Riesz for conjugate functions can be stated as follows:

THEOREM 1.3. *To each p such that* $1 < p < \infty$ *there corresponds a constant* A_p *such that the inequality*

$$\|\operatorname{Im} f(z)\|_p \leqslant A_p \|\operatorname{Re} f(z)\|_p$$

holds for every holomorphic function f in U with $\operatorname{Im} f(0) = 0$.

And the Kolmogorov's theorem for conjugate functions may be stated as follows:

THEOREM 1.4. *To each p such that* $0 < p < 1$ *there corresponds a constant* A_p *such that the inequality*

$$\|\operatorname{Im} f(z)\|_p \leqslant A_p \|\operatorname{Re} f(z)\|_1$$

holds for every holomorphic function f in U, with $\operatorname{Im} f(0) = 0$.

There appear to be no further generalizations of the above two theorems to the solutions of elliptic systems of equations other than Cauchy-Riemann equations.

We give two methods to establish the generalized Riesz and Kolmogorov theorems. The first method is based on the integral representation (3.1) which we believe can be extended to many other systems. The second method is based on the well-known Similarity Principle which is however only good for the system (1.1).

This paper is organized as follows. §2 describes the notations, complex form of equation (1.1), the well-known properties of Cauchy type integrals that are involved, as well as certain integral operators on the xy plane. §3 states the integral representation for $v(z)$ of a solution $(u(z), v(z))$ of (1.1) in terms of $u(z)$. This representation will enable us to prove a number of theorems generalizing properties of holomorphic functions. In this section, we also state the well-known Similarity Principle. §4 establishes the generalized Privoloff theorem for (1.1). §5 extends the Schwarz reflection principle to (1.1). §6 extends the M. Riesz's theorem for conjugate functions to (1.1). §7 gives the generalized Kolmogorov's theorem for (1.1).

The contribution of this work is threefold.

(1) It generalizes the Schwarz and Privoloff theorems to the elliptic system of differential equations (1.1) without any assumptions of $\nu(z)$ on the boundary.

(2) It extends the M. Riesz theorem and Kolmogorov theorem for conjugate functions to the solutions of (1.1).

(3) It points out that the intimacy of the real part and imaginary part of a holomorphic function can be expected as well for the solutions of elliptic system of equations.

2. Notations, complex form of (1.1) and general remarks.

A. *Notations.* ∂A is the boundary of a set A.

$\overline{A} = \partial A \cup A$.

C is the complex plane (or xy plane).

$C^k(\overline{G})$, where $\overline{G}$ is a closed domain in C, is the space of k times continuously differentiable complex valued functions in $\overline{G}$, $0 \leqslant k \leqslant \infty$.

$C(\overline{\Omega})$ is sometimes used as a shorter notation for $C^0(\overline{\Omega})$.

$C(f, \overline{G})$ is the norm of a function f in G and is defined according to the formula

$$C(f, \overline{G}) \equiv C(f) = \max_{z \in \overline{G}} |f(z)|, \qquad z = x + iy.$$

$C^m(f, \overline{G})$ is the norm of a function f in $C^m(\overline{G})$ and is defined according to the formula

$$C^m(f, \overline{G}) \equiv C^m(f) = \sum_{k=0}^{m} \sum_{l=0}^{k} C\left(\frac{\partial^k f}{\partial x^{k-l} \partial y}, \overline{G}\right).$$

$H_\alpha(\overline{G})$ is the space of all Hölder continuous functions with index α, $0 < \alpha \leqslant 1$, in $\overline{G}$.

$H(f)$[or $H(f, \alpha)$, or else by $H(f, \alpha, \overline{G})$] is the Hölder constant of a function f in $H_\alpha(\overline{G})$ and is defined by

$$H(f) \equiv H(f, \alpha, \overline{G}) = \sup_{z_1, z_2 \in \overline{G}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}.$$

$C_\alpha(\overline{G})$ is the space of bounded Hölder continuous functions in $H_\alpha(\overline{G})$ such that

$$C_\alpha(f, \overline{G}) = C_\alpha(f) = C(f, \overline{G}) + H(f, \alpha, \overline{G}) < \infty.$$

$C_\alpha^m(\overline{G})$ is the space of functions in $C^m(\overline{G})$ such that

$$\frac{\partial^m f}{\partial x^{m-k} \partial y^k} \in C_\alpha(\overline{G}) \qquad (k = 0, 1, \ldots, m), \; 0 < \alpha \leqslant 1.$$

$C_\alpha^m(f)$ [or $C_\alpha^m(f, G)$] is the norm of a function f belonging to $C_\alpha^m(\overline{G})$ and is defined by the formula

$$C_\alpha^m(f) \equiv C_\alpha^m(f, \overline{G}) = C^m(f) + \sum_{k=0}^{m} H\left(\frac{\partial^m f}{\partial x^{m-k} \partial y^k}, \alpha, \overline{G}\right).$$

$C^k(\Gamma)$, where Γ is a rectifiable simple Jordan curve as defined in Definition 2.1.

$C_\alpha^k(\Gamma)$ is defined in Definition 2.1.

$C^m(f, \Gamma)$ is defined in Definition 2.2.

$C_\alpha^m(f, \Gamma)$ is defined in Definition 2.3.

$L_p(\Omega)$ is the space of measurable functions in Ω such that the norm

$$\|u\|_{L_p} = \left|\int |u|^p \, dx \, dy\right|^{1/p} < \infty, \quad 0 < p < \infty.$$

U is a notation for $\{|z| < x\}$.

$M_p(f; r)$, where f is a continuous function in U, is defined according to the formula

$$M_p(f; r) = \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}, \qquad 0 < p < \infty.$$

$\|f\|_p$, where f is a continuous function in U, is defined according to the formula

$$\|f\|_p = \sup_{0 < r < 1} M_p(f; r).$$

$\operatorname{Re} f(z)$ = the real part of $f(z)$, $\operatorname{Im} f(z)$ = the imaginary part of $f(z)$.

B. *Complex form of* (1.1).

It is convenient to investigate system (1.1) in the complex form. Introducing the notation

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right),$$

we write system (1.1) in the form of one complex equation

$$\partial w/\partial \bar{z} = Aw + B\bar{w} + F, \tag{2.1}$$

where

$$w = u + iv, \qquad A = \tfrac{1}{4}(a + d + ic - ib),$$
$$B = \tfrac{1}{4}(a - d + ic + ib), \qquad F = \tfrac{1}{2}(f + ig). \tag{2.2}$$

Conversely, separating in (2.1) the real and imaginary parts we return to system (1.1).

If $A = B = F = 0$ we obtain the equation

$$\partial w/\partial \bar{z} = 0 \tag{2.3}$$

which constitutes a system of two Cauchy-Riemann equations. Thus the solution $w = u + iv$ of (2.3) is an ordinary analytic function of the complex variable $z = x + iy$.

C. *Properties of an integral operator on G.*

Consider a double integral over a bounded domain G, with Cauchy kernel

$$g(z) = -\frac{1}{\pi}\iint_G \frac{f(\xi,\eta)}{\zeta - z}\,d\xi\,d\eta \tag{2.4}$$

where $f(x,y)$ is defined on G.

THEOREM 2.1. *If $f \in L_p(G)$, $p > 2$, then the following relations hold:*

(1) $(\partial/\partial\bar{z})g = f$,

(2) $|g(z)| \leqslant M_1\|f\|_{L_p}$,

(3) $|g(z_1) - g(z_2)| \leqslant M_2\|f\|_{L_p}|z_1 - z_2|^{1-2/p}$,

where z_1 and z_2 are arbitrary points of the plane, and M_1, M_2 are constants, M_1 depending on p and G, while M_2 depends on p only.

PROOF. See Vekua [12, p. 38].

THEOREM 2.2. *If $f(z) \in C_\alpha^m(\bar{U})$, $0 < \alpha < 1$, then the function*

$$h(z) = \frac{1}{\pi}\iint_U \frac{f(\zeta)}{\zeta - z}\,d\xi\,d\eta$$

belongs to $C_\alpha^{m+1}(\bar{U})$, where $\bar{U} = \{|z| \leqslant 1\}$.

PROOF. See Vekua [12, p. 56].

THEOREM 2.3. *If $f \in L_1(G)$, then $g(z)$ in (2.4) exists almost everywhere and belongs to an arbitrary class $L_p(G_*)$ where p is an arbitrary number satisfying the condition $1 \leqslant p < 2$ and G_* is an arbitrary bounded domain of the plane.*

PROOF. See Vekua [12, p. 28].

THEOREM 2.4. *Let G be a bounded domain with a piecewise smooth boundary* ∂G. *If* $w \in C(G)$, $w_{\bar z} \in L_p(\overline{G})$, $p > 2$, *then*

$$w(z) = f(z) - \frac{1}{\pi}\iint_G \frac{w_{\bar\zeta}(\zeta)}{\zeta - z} d\xi\, d\eta, \qquad \zeta = \xi + i\eta, \tag{2.5}$$

where $f(z)$ *is a function holomorphic in G. Furthermore, if* $w \in C(G \cup \partial G)$ *and* $w_{\bar z} \in L_p(G)$, $p > 2$, *then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial G} \frac{w(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi}\iint_G \frac{w_{\bar\zeta}(\zeta)}{\zeta - z} d\xi\, d\eta.$$

PROOF. See Vekua [12, pp. 34 and 41].

THEOREM 2.5. *Let A, B, F in* (2.1) *be bounded measurable on U. If w is a solution of* (2.1) *in U,* $w \in C(U)$, $w_{\bar z} \in L_p(U)$, $p > 2$, *then* $w \in C_\alpha(\overline{U}_r)$, $0 < \alpha < 1$, *where* $\overline{U}_r = \{|z| \leqslant r < 1\}$.

PROOF. Let us consider two subdomains U_r and U_{r_1} of the domain U, $U_r \subset \overline{U}_r \subset U_{r_1} \subset \overline{U}_{r_1} \subset U$. Then, according to Theorem 2.4 we have

$$w(z) = f_{r_1}(z) - \frac{1}{\pi}\iint_{U_{r_1}} \frac{Aw + B\overline{w} + F}{\zeta - z} d\xi\, d\eta = f_{r_1}(z) - Tw$$

for $|z| < r_1$, where $f_{r_1}(z)$ is a function holomorphic in U_{r_1} and $Tw \in C_\alpha(U_{r_1})$ (Theorem 2.1). Since $f_{r_1}(z)$ is holomorphic in U_{r_1}, we have $f_{r_1} \in C_\alpha(\overline{U}_r)$, and the lemma follows.

THEOREM 2.6. *Let A, B, F in* (2.1) *belong to* $C^m_\alpha(\overline{U})$. *If w is a solution of* (2.1) *in* U, $w \in C(U)$, $w_{\bar z} \in L_p(U)$, $p > 2$. *Then* $w \in C^{m+1}_\alpha(\overline{U}_r)$, $0 < \alpha < 1$, *where* $U_r = \{|z| < r < u\}$.

PROOF. Let us consider $m + 2$ subdomains $U_r, U_{r_1}, \ldots, U_{r_{m+1}}$ such that

$$U_r \subset \overline{U}_r \subset U_{r_1} \subset \overline{U}_{r_1} \subset \cdots \subset U_{r_m} \subset \overline{U}_{r_m} \subset U_{r_{m+1}} \subset \overline{U}_{r_{m+1}} \subset U.$$

Then, according to Theorem 2.4, we have

$$w(z) = f_{r_i}(z) - \frac{1}{\pi}\iint_{U_{r_i}} \frac{Aw + B\overline{w} + F}{\zeta - z} d\xi\, d\eta = f_{r_i}(z) - T_i w$$

for $z \in U_{r_i}$, $i = 1, \ldots, m + 1$, where $f_{r_i}(z)$ is a function holomorphic in U_{r_i}.

According to Theorem 2.1, $T_{m+1} w \in C_\alpha(\overline{U}_{r_{m+1}})$. Since $f_{r_{m+1}}$ is holomorphic in $U_{r_{m+1}}$, $f_{r_{m+1}} \in C_\alpha(\overline{U}_m)$. Hence $w \in C_\alpha(\overline{U}_m)$. Since $Aw + B\overline{w} + F \in C_\alpha(\overline{U}_m)$, by Theorem 2.2, $T_m w \in C^1_\alpha(\overline{U}_{r_m})$. And since f_{r_m} is holomorphic in U_{r_m}, $f_{r_m} \in C^1_\alpha(\overline{U}_{r_{m-1}})$. Therefore, $w \in C^1_\alpha(\overline{U}_{r_{m-1}})$.

Repeating this reasoning we obtain $w \in C_\alpha^{m+1}(\overline{U}_r)$.

D. *Properties of Cauchy type integrals.* The following two lemmas are concerned with some fundamental properties of an integral of the Cauchy type on the $|z| = 1$, which can be found in Gakhov [5], Neri [9] or Muskhelishvili [8].

LEMMA 2.1. *If $\phi(t) \in L_p$, $p > 1$, then the singular integral*

$$S\phi = \frac{1}{\pi i}\int_{|z|=1}\frac{\phi(\tau)}{\zeta - t}d\tau \tag{2.6}$$

also belongs to L_p and has the following estimate

$$\|S\phi\|_{L_p} \leqslant M_p\|\phi\|_{L_p} \tag{2.7}$$

where M_p depends only on p.

PROOF. See Gakhov [5] or Neri [9].

Let there be given a function $f(z)$ of the point $z \in \Gamma$ on a rectifiable simple Jordan curve Γ. This function may be regarded as a function of the length of the arc s, i.e., $f(z(s)) = f(s)$.

DEFINITION 2.1. A function $f(z)$ defined on Γ is said to belong to the class $C^m(\Gamma)$ if $f(s)$ $(= f(z(s)))$ and all its derivatives up to the mth order are continuous on the arc $0 \leqslant s \leqslant l$. If, moreover, $f^{(m)}(s)$ satisfies the Hölder condition with an index α, $0 < \alpha \leqslant 1$, then it will be said that $f \in C_\alpha^m(\Gamma)$.

DEFINITION 2.2. The norm $C^m(f,\Gamma)$ of a function $f \in C^m(\Gamma)$ is defined by the formula

$$C^m(f,\Gamma) = \sum_{k=0}^{m} C\left(\frac{d^kf}{ds^k}, \Gamma\right) \tag{2.8}$$

where

$$C(f,\Gamma) = \max_{t\in\Gamma}|f(t)|. \tag{2.9}$$

DEFINITION 2.3. The norm $C_\alpha^m(f,\Gamma)$ of a function $f \in C_\alpha^m(\Gamma)$ is defined by the formula

$$C_\alpha^m(f,\Gamma) = C^m(f,\Gamma) + H\left(\frac{d^mf}{ds^m}, \Gamma, \alpha\right) \tag{2.10}$$

where

$$H(f,\Gamma,\alpha) = \sup_{t_1,t_2\in\Gamma}\frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha}. \tag{2.11}$$

LEMMA 2.2. *Let U denote $\{|z| < 1\}$. If $\psi(z)$ satisfies the Hölder condition*

$$|\psi(e^{i\theta_1}) - \psi(e^{i\theta_2})| < k|e^{i\theta_1} - e^{i\theta_2}|^\alpha$$

then the function

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{\psi(t)}{t - z} dt \tag{2.12}$$

can be defined as a Hölder continuous function in $\overline{U}$ and satisfies

(i) $$|\phi(z_1) - \phi(z_2)| \leqslant CK|z_1 - z_2|^\alpha$$

for $z_1, z_2 \in \overline{U}$, and

(2.13) (ii) $$\phi(z) = \frac{1}{2}\psi(z) + \frac{1}{2\pi i}\int_{\partial U} \frac{\psi(t)}{t - z} dt$$

for $|z| = 1$, where the singular integral

$$\int_{\partial U} \frac{\psi(t)}{t - z} dt, \qquad |z| = 1, \tag{2.14}$$

is understood in the sense of the principal value.

Furthermore, if $\psi(t) \in C_\alpha^m(\partial U)$, then $\phi(t) \in C_\alpha^m(\overline{U})$, and satisfies

$$C_\alpha^m(\phi, \overline{U}) \leqslant MC_\alpha^m(\psi, \partial U) \tag{2.15}$$

where M is a constant independent of ψ.

PROOF. See Vekua [12] or Muskhelishvili [8].

LEMMA 2.3 (KOLMOGOROV). *Let B be a bounded set in $(-\infty, \infty)$. If $f \in L(-\infty, \infty)$, then the function*

$$\tilde{f}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x - t} dt \tag{2.16}$$

belongs to $L^{1-\alpha}(B)$, whenever $0 < \alpha < 1$, and

$$\left(\int_B |\tilde{f}(x)|^{1-\alpha} dx\right)^{1/(1-\alpha)} \leqslant A_\alpha \|f\|_L \tag{2.17}$$

where A_α is independent of f, and the singular integral is understood in the sense of the principal value.

PROOF. See Neri [9, p. 79].

LEMMA 2.4. *If $f(t) \in L(\partial U)$, then the function*

$$\tilde{f}(z) = \int_{\partial U} \frac{f(t)}{t - z}\, dt$$

belongs to $L_p(\partial U)$, *whenever* $0 < p < x$, *and there exists a constant* A_p *such that*

$$\|\tilde{f}(z)\|_{L_p} \leqslant A_p \|f\|_L \tag{2.18}$$

where $\| \ \|_{L_p}$ *denotes the* L_p *norm on* ∂U.

PROOF. Let $z = e^{i\phi}$, $t = e^{i\theta}$, then $dt = ie^{i\theta} d\theta$, and

$$\frac{dt}{t - z} = \frac{ie^{i\theta} d\theta}{e^{i\theta} - e^{i\phi}} = \frac{1}{2} \cot \frac{\theta - \phi}{2} d\theta + \frac{i}{2} d\theta. \tag{2.19}$$

The function $\cot \psi$ has a simple pole at $\psi = 0, \pi, -\pi$; hence we can express the function

$$\frac{1}{2} \cot \frac{\theta - \phi}{2} = \frac{1}{\theta - \phi} + \frac{1}{\theta - \phi - 2\pi} + \frac{1}{\theta - \phi + 2\pi} + P(\theta - \phi) \tag{2.20}$$

where $P(\psi)$ is an analytic function for a real variable ψ, whenever $-4\pi < \psi < 4\pi$.

Therefore

$$\begin{aligned} f(z) = f(e^{i\phi}) &= \int_{-\pi}^{\pi} \frac{f(e^{i\theta})\, d\theta}{\theta - \phi} + \int_{-\pi}^{\pi} \frac{f(e^{i\theta})\, d\theta}{\theta - \phi - 2\pi} \\ &+ \int_{-\pi}^{\pi} \frac{f(e^{i\theta})\, d\theta}{\theta - \phi + 2\pi} + \int_{-\pi}^{\pi} f(e^{i\theta}) P(\theta - \phi)\, d\theta \\ &+ \frac{i}{2} \int_{-\pi}^{\pi} f(e^{i\theta})\, d\theta. \end{aligned} \tag{2.21}$$

By Lemma 2.3, there exists a constant B_p:

$$\left\| \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{\theta - \phi} d\theta \right\|_{L_p} \leqslant B_p \|f\|_L, \tag{2.22}$$

$$\left\| \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{\theta - \phi - 2\pi} d\theta \right\|_{L_p} \leqslant B_p \|f\|_L, \tag{2.23}$$

and

$$\left\| \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{\theta - \phi + 2\pi} d\theta \right\|_{L_p} \leqslant B_p \|f\|_L \tag{2.24}$$

where $\| \ \|_{L_p}$ denotes the L_p norm for $(-\pi, \pi)$, $0 \leqslant p < 1$. Following the above inequalities, the formula (2.21) gives the lemma.

3. Basic Lemma and Similarity Principle. We shall first derive an integral representation of a solution $w = u + iv$ of (2.1) in terms of u. This representation is known and proved in Vekua [12, p. 202], but in a qualitative form only. We give the proof here.

LEMMA 3.1 (BASIC LEMMA). *Let G be a bounded domain, and let the functions A, B, F be bounded measurable in G. If $w(z)$ is a solution of (2.1) in G, $w(z) \in C(G)$, $w_{\bar z} \in L_p(G)$, and $u(z) \in L_p(G)$, $p > 2$, then*

$$w(z) = \frac{f(z)}{\mu(z)} - \frac{1}{\pi\mu}\iint_G \frac{2B\mu u + \mu F}{\zeta - z}\,d\xi\,d\eta \tag{3.1}$$

for $z \in G$, where $f(z)$ is a holomorphic function in G, and

$$\mu(z) = \exp\left[\frac{1}{\pi}\iint_G \frac{A - B}{\zeta - z}\,d\xi\,d\eta\right]. \tag{3.2}$$

PROOF. Write the equation (2.1) in the form

$$w_{\bar z} = (A - B)w + 2Bu + F. \tag{3.3}$$

We want to choose a function $\mu \neq 0$, if possible, so that

$$\mu(z)[w_{\bar z} - (A - B)w] = \frac{\partial}{\partial \bar z}[\mu(z)w(z)]. \tag{3.4}$$

By Theorem 2.1, we see

$$\mu(z) = \exp\left[\frac{1}{\pi}\iint_G \frac{A - B}{\zeta - z}\,d\xi\,d\eta\right].$$

Multiplying (3.3) by $\mu(z)$, we see

$$\frac{\partial}{\partial \bar z}[\mu(z)w(z)] = 2B\mu u + \mu F.$$

Hence, by Theorem 2.4, we have

$$\mu(z)w(z) = f(z) - \frac{1}{\pi}\iint_G \frac{2B\mu u + \mu F}{\zeta - z}\,d\xi\,d\eta, \tag{3.5}$$

where $f(z)$ is a function holomorphic in G. This completes the proof.

LEMMA 3.2 (SIMILARITY PRINCIPLE). *Let $A, B \in L_p(G)$, $p > 2$. Let $w(z)$ be a solution of*

$$w_{\bar z} = A(z)w(z) + B(z)\overline{w(z)} \tag{3.6}$$

in G. Also let

$$g(z) = A(z) + B(z)\frac{\overline{w(z)}}{w(z)}, \qquad \text{if } w(z) \neq 0,\ z \in G,$$
$$(3.7) \qquad = A(z) + B(z), \qquad \text{if } w(z) = 0,\ z \in G.$$

Then the function

$$(3.8) \qquad f(z) = w(z)e^{-\omega(z)},$$

where

$$(3.9) \qquad \omega(z) = \frac{1}{\pi}\iint_G \frac{g(\zeta)}{\zeta - z}\,d\xi\,d\eta,$$

is holomorphic in G.

PROOF. See Vekua [12, p. 144].

LEMMA 3.3. *Let A, B, F of* (2.1) *belong to* $L_p(G)$, $p > 2$. *Then there exists a particular solution* $w_1(z)$ *of* (2.1) *in G, which may be taken in the form*

$$w_1(z) = -\frac{1}{\pi}\iint_G \Omega_1(z,\zeta)F(\zeta)\,d\xi\,d\eta - \frac{1}{\pi}\iint_G \Omega_2(z,\zeta)\overline{F(\zeta)}\,d\xi\,d\eta$$

where Ω_1, Ω_2 *depend only on A, B and are given explicitly in Vekua* [12, *p.* 187]. *Furthermore,* $w_1(z) \in C_{(p-2)/p}(\overline{G})$.

4. Generalized Privoloff theorem. In the present section we shall establish the Privoloff type theorem for equation (2.1). We shall assume that the functions A, B, F in (2.1) are Hölder continuous in $\{|z| \leqslant 1\}$.

We first state some well-known results which are concerned with the Hilbert boundary value problem for the unit disk. By the Hilbert boundary value problem we understand the following problem. It is required to find in $|z| < 1$ a holomorphic function $f(z) = u + iv$ which is continuous for $|z| \leqslant 1$, and satisfies the linear relation

$$(4.1) \qquad a(s)u(s) + b(s)v(s) = c(s), \qquad \sqrt{a^2 + b^2} = 1,$$

on $|z| = 1$, where a, b, c are Hölder continuous on $|z| = 1$.

DEFINITION 4.1. By the index of the function $a(s) + ib(s)$ with respect to the circle $|z| = 1$, we understand the increment of its argument, in traversing the circle in the positive direction, divided by 2π. The index of $a(s) + ib(s)$ on $|z| = 1$ can be written in the form

$$(4.2) \qquad \chi = \operatorname{Ind}(a(s) + ib(s)) = \frac{1}{2\pi}[\arg(a(s) + ib(s))]_{|z|=1}.$$

Denoting

(4.3)
$$\nu(z) = \frac{1}{2\pi}\int_0^{2\pi}\left[\tan^{-1}\frac{b(\sigma)}{a(\sigma)} - \chi\sigma\right]\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma$$

and without ambiguity $\nu(\theta)$ is understood as $\nu(z(\theta)) = \nu(z)$. We find that the solution of the Hilbert problem (4.1) for the case $\chi = 0$ is given by the formula

(4.4)
$$f(z) = e^{i\nu(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu(\sigma)}C(\sigma)\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma + i\beta_0\right],$$

where β_0 is an arbitrary real constant, and $C(\sigma)$ is given in (4.1).

LEMMA 4.1. *Let G be a real valued function belonging to $C_\alpha^m(\bar{U})$, and H be a real valued function belonging to $C(\bar{U}) \cap C_\alpha^m(\partial U_r)$, $0 < \alpha < 1$, for every r, $0 < r \leqslant 1$, where $U_r = \{|z| < r\}$. If $f(z) = u(z) + iv(z)$ is holomorphic in $|z| < 1$, and satisfies the condition*

(4.5)
$$\operatorname{Re}(f(z)e^{-iG(z)}) = H(z)$$

for $|z| < 1$, then $f(z)$ can be defined as a function in $C_\alpha^m(\bar{U})$.

PROOF. It is clear that the index of e^{+iG} on $|z| = r \leqslant 1$ is equal to 0.
For $r < 1$, we have, by formula (4.4),

(4.6)
$$f(z) = e^{i\nu_r(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu_r(\sigma)}H(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}\,d\sigma + i\beta_0(r)\right],$$

for $|z| < r$, where

(4.7)
$$\nu_r(z) = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}\,d\sigma$$

and $\beta_0(r)$ can be determined by the formula

(4.8)
$$\beta_0(r) = -if(0)e^{i\nu_r(0)} + \frac{i}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu_r(\sigma)}H(re^{i\sigma})\,d\sigma.$$

Since $G(re^{i\sigma})$ and $H(re^{i\sigma})$ are continuous in $r \leqslant 1$, $0 \leqslant \sigma \leqslant 2\pi$, we may let $r \to 1$ obtaining

(4.9)
$$f(z) = e^{i\nu(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu(\sigma)}H(e^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma + i\beta_0\right]$$

for $|z| < 1$, where

(4.10)
$$\nu(z) = \frac{1}{2\pi}\int_0^{2\pi} G(e^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma$$

and

$$\beta_0 = -if(0)e^{i\nu(0)} + \frac{i}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu(\sigma)} H(e^{i\sigma})\, d\sigma. \tag{4.11}$$

Using the identity

$$\frac{e^{i\sigma} + z}{e^{i\sigma} - z} d\sigma = \left(-1 + \frac{2e^{i\sigma}}{e^{i\sigma} - z}\right) d\sigma = \frac{2}{i}\frac{d\zeta}{\zeta - z} - d\sigma \tag{4.12}$$

for $|\zeta| = 1$, (4.9) then gives

$$\begin{aligned} f(z) = e^{i\nu(z)}\Big[&\frac{2}{2\pi i}\int_{\partial U} \frac{e^{i\operatorname{Im}\nu(\zeta)} H(\zeta)}{\zeta - z} d\zeta \\ &- \frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu(\sigma)} H(e^{i\sigma})\, d\sigma + i\beta_0\Big] \end{aligned} \tag{4.13}$$

for $|z| < 1$, where

$$\nu(z) = \frac{1}{\pi i}\int_{\partial U} \frac{G(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi}\int_0^{2\pi} G(e^{i\sigma})\, d\sigma. \tag{4.14}$$

By Lemma 2.2, $\nu(z) \in C_\alpha^m(\overline{U})$ and the function

$$\int_{\partial U} \frac{e^{i\operatorname{Im}\nu(\zeta)} H(\zeta)}{\zeta - z} d\zeta \tag{4.15}$$

belongs to $C_\alpha^m(\overline{U})$. The assertion follows from these statements.

THEOREM 4.1. *Let A, B, F of* (2.1) *belong to $C_\alpha^m(\overline{U})$. If $w(z) = u(z) + iv(z)$ is a solution of* (2.1) *in U, and $u(z) \in C(\overline{U}) \cap C_\alpha^{m+1}(\partial U)$, then $w(z) \in C_\alpha^{m+1}(\overline{U})$, $0 < \alpha < 1$.*

PROOF. By Lemma 3.1, we have

$$u(z) = \operatorname{Re}\frac{f(z)}{\mu(z)} - \operatorname{Re}\frac{1}{\pi\mu(z)}\iint_U \frac{2B\mu u + \mu F}{\zeta - z} d\xi\, d\eta \tag{4.16}$$

for $|z| < 1$. Setting

$$e^{K - iG} = \frac{1}{\mu(z)} = \exp\left[\frac{1}{\pi}\iint_U \frac{B - A}{\zeta - z} d\xi\, d\eta\right], \tag{4.17}$$

we obtain

$$\operatorname{Re}(f(z)e^{-iG(z)}) = \beta(z) \tag{4.18}$$

where

$$\beta(z) = e^{-K(z)}\left\{u(z) + \operatorname{Re}\frac{1}{\pi\mu(z)}\iint_U \frac{2B\mu u + \mu F}{\zeta - z}d\xi\, d\eta\right\}. \tag{4.19}$$

By Theorem 2.8, $w \in C_\alpha^{m+1}(U_r)$, $U_r = \{|z| < r < 1\}$. Hence $u \in C(\overline{U}) \cap C_\alpha^{m+1}(\partial U_r) \cap C_\alpha^{m+1}(\partial U)$. According to Theorem 2.2, $\mu(z) \in C_\alpha^{m+1}(\overline{U})$, also because of Theorem 2.1 we have

$$g(z) = \frac{1}{\pi\mu(z)}\iint_U \frac{2B\mu u + \mu F}{\zeta - z}d\xi\, d\eta \in C_\alpha(\overline{U}), \tag{4.20}$$

and hence $\beta(z) \in C_\alpha(\partial U) \cap C_\alpha(\partial U_r)$. Therefore, by Lemma 4.1, $f(z) \in C_\alpha(\overline{U})$. Then using (4.20) and (3.1), we obtain $w(z) \in C_\alpha(\overline{U})$.

Next we want to prove that $w(z) \in C_\alpha^1(\overline{U})$. Since $u \in C_\alpha(\overline{U}) \cap C_\alpha^{m+1}(\partial U) \cap C_\alpha^{m+1}(\partial U_r)$ by Theorem 2.2, $g(z) \in C_\alpha^1(\overline{U})$, and hence $\beta(z) \in C_\alpha^1(\partial U) \cap C_\alpha^1(\partial U_r)$. We obtain, in view of Lemma 4.1, $f(z) \in C_\alpha^1(\overline{U})$, so that $w(z) \in C_\alpha^1(\overline{U})$.

By continuing a similar reasoning we conclude $w(z) \in C_\alpha^m(\overline{U})$.

For the case $m = 0$, we will have the following refinement.

THEOREM 4.2. *Let A, B, F of* (2.1) *be bounded measurable in* $|z| \leqslant 1$. *Let* $w(z) = u(z) + iv(z)$ *be a solution of* (2.1) *in* $|z| < 1$. *If* $u(z)$ *is continuous in* $|z| \leqslant 1$, $|u(e^{i\theta})| \leqslant M$ *and satisfies*

$$|u(e^{i\theta_1}) - u(e^{i\theta_2})| \leqslant K|e^{i\theta_1} - e^{i\theta_2}|^\alpha \tag{4.21}$$

then $w(z)$ *is continuous in* $|z| \leqslant 1$ *and satisfies*

$$|w(z_1) - w(z_2)| \leqslant CK|z_1 - z_2|^\alpha \tag{4.22}$$

where the constant C depends only on α, $0 < \alpha < 1$ *and* A, B, F, M.

PROOF. The solution $w(z)$ of (2.1) is representable in the form $w = w_0 + w_1$ where w_0 is a solution of the homogeneous equation

$$\partial_{\bar z} w_0 = Aw_0 + B\overline{w}_0 \tag{4.23}$$

and consequently it is given by the realation (Lemma 3.2)

$$w_0(z) = f(z)e^{\omega(z)}, \qquad \omega(z) = \frac{1}{\pi}\iint_U \left(A + B\frac{\overline{w}_0}{w_0}\right)\frac{d\xi\, d\eta}{\zeta - z}, \tag{4.24}$$

w_1 is a particular solution of the nonhomogeneous equation (2.1), and belongs to $C_\alpha(\overline{U})$ (Lemma 3.3).

The function $\omega \in C_\alpha(\overline{U})$ and the function $f(z)$ is holomorphic in U.

Set

$$\zeta(z) = \operatorname{Re} w(z), \qquad G(z) = -\operatorname{Im} w(z); \tag{4.25}$$

then $f(z)$ satisfies the condition

$$\operatorname{Re}(f(z)e^{-iG(z)}) = H(z) \tag{4.26}$$

for $z \in G$, where

$$H(z) = u(z)e^{-\zeta(z)} - \operatorname{Re} e^{-\zeta(z)}w_1(z). \tag{4.27}$$

For $r < 1$, we have (by formula (4.4))

$$f(z) = e^{i\nu_r(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu_r(\sigma)}H(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}d\sigma + i\beta_0(r)\right], \tag{4.28}$$

for $|z| < r$, where

$$\nu_r(z) = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}d\sigma \tag{4.29}$$

and $\beta_0(r)$ can be determined by the formula

$$\beta_0(r) = -if(0)e^{i\nu_r(0)} + \frac{i}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu_r(\sigma)}H(re^{i\sigma})\,d\sigma. \tag{4.30}$$

Since $G(re^{i\sigma})$ and $H(re^{i\sigma})$ are continuous in $r \leqslant 1$, $0 \leqslant \sigma \leqslant 2\pi$, we may let $r \to 1$ obtaining

$$f(z) = e^{i\nu(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu(\sigma)}H(e^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}d\sigma + i\beta_0\right] \tag{4.31}$$

for $|z| < 1$, where

$$\nu(z) = \frac{1}{2\pi}\int_0^{2\pi} G(e^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}d\sigma \tag{4.32}$$

and

$$\beta_0 = -if(0)e^{i\nu(0)} + \frac{i}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu(\sigma)}H(e^{i\sigma})\,d\sigma. \tag{4.33}$$

It follows from (4.31) that (Lemma 2.2) $|f(z)| \leqslant C_1 M$, and

$$|f(z_1) - f(z_2)| \leqslant C_1 K|z_u - z_2|^\alpha. \tag{4.34}$$

Therefore, it follows from (4.24)

$$|w_0(z_1) - w_0(z_2)| \leqslant C_2 K|z_1 - z_2|^\alpha. \tag{4.35}$$

Hence the desired result follows from the relation $w = w_0 + w_1$.

In respect of the Riemann-Hilbert type boundary value problem Vekua [12] has established the following theorem.

THEOREM 4.3 (VEKUA [12, p. 223]). *Let* A, B, F *of* (2.1) *be bounded measurable on* $\overline{U}$. *If* $w(z) = u + iv$ *is a solution of* (2.1) *in* U, *continuous in* $\overline{U}$ *and satisfies the boundary condition*

$$\alpha u + \beta v \equiv Re[\overline{\lambda(z)}w] = \nu(z) \quad (on\ \partial U),$$

$\lambda = \alpha + i\beta$, $|\lambda(z)| = 1$, *where* $\lambda(z)$ *and* $\nu(z) \in C_\mu(\partial U)$, $0 < \mu < 1$, *then* $w(z) \in C_\mu(\overline{U})$.

Since Vekua's method depends on the Similarity Principle (Lemma 3.2), therefore it is difficult to apply his method to study the differential properties of $w(z)$ in $\overline{U}$ under the differential boundary conditions.

We shall introduce an elementary method to study this problem.

THEOREM 4.4. *Let* A, B, F *of* (2.1) *belong to* $C_\mu^m(\overline{U})$, $0 < \mu < 1$. *If* $w(z) = u(z) + iv(z)$ *is a solution of* (2.1) *in* U, *continuous in* $\overline{U}$ *and satisfies the boundary condition*

$$\alpha u + \beta v \equiv \mathrm{Re}[\overline{\lambda(z)}w] = \nu(z) \quad (on\ \partial U), \tag{4.36}$$

where $\lambda = \alpha + i\beta$, $|\lambda(z)| = 1$, $\lambda(z)$ *and* $\nu(z) \in C_\mu^{m+1}(\partial U)$, $0 < \mu < 1$, *then* $w(z) \in C_\mu^{m+1}(\overline{U})$.

PROOF. The solution $w(z)$ is representable in the form (Theorem 2.4)

$$w(z) = f(z) - \frac{1}{\pi}\iint_U \frac{Aw + B\overline{w} + F}{\zeta - z}\, d\xi\, d\eta = f(z) - g(z) \tag{4.37}$$

where $f(z)$ is a function holomorphic in U, continuous in $\overline{U}$, and satisfies the boundary condition

$$\mathrm{Re}\,[\overline{\lambda(z)}f(z)] = \nu_0(z) \quad (\text{on } \partial U), \tag{4.38}$$

where

$$\nu_0(z) = \nu(z) + \mathrm{Re}\,[\overline{\lambda(z)}g(z)].$$

The function $g(z) \in C_\mu(\overline{U})$ (Theorem 2.1); hence $\nu_0(z) \in C_\mu(\partial U)$. We want to prove that the solution $f(z)$ of the Riemann-Hilbert problem (4.38) belongs to the class $C_\mu(\overline{U})$.

The function $\lambda(z)$ can be represented on ∂U in the form

$$\overline{\lambda(z)} = z^{-n} e^{\chi(z)} e^{-p(z)} \tag{4.39}$$

where $\chi(z) = p + iq$ is a function holomorphic in $|z| < 1$, the imaginary part of which on $|z| = 1$ is given by $q = -\arg \nu_0(z) + n \arg z$, the integer n is so chosen that every branch of $q(z)$ is a single valued function on $|z| = 1$. The function $\chi(z)$ may be constructed by means of the Schwarz integral

$$\chi(z) = \frac{1}{2\pi}\int_{\partial U} q(t)\frac{t+z}{t-z}\frac{dt}{t}. \tag{4.40}$$

Since $q \in C_\mu(\partial U)$, $\chi(z) \in C_\mu(\overline{U})$ (Lemma 2.2). Moreover,

$$C_\mu(\chi, \overline{U}) \leqslant M_\mu C_\mu(q, \partial U), \qquad M_\mu = \text{constant}.$$

Introducing the expression (4.39) into the boundary condition (4.38) we obtain

$$\operatorname{Re}[z^{-n}e^{\chi(z)}f(z)] = \nu_1(z), \qquad \nu_1(z) = \nu_0 e^{p(z)}. \tag{4.41}$$

Evidently, $\nu_1 \in C_\mu(\partial U)$. If $n < 0$, (4.41) implies that

$$f(z) = \frac{z^n e^{-\chi(z)}}{2\pi i}\int_{\partial U} \nu_1(t)\frac{t+z}{t-z}\frac{dt}{t} + iC_0 z^n e^{-\chi(z)}, \tag{4.42}$$

where C_0 is a real constant. Hence, in view of the continuity of $f(z)$ we have

$$C_0 = 0, \quad \int_0^{2\pi} \nu_1(e^{i\theta})e^{-ki\theta}\,d\theta = 0 \quad (k = 0, \ldots, -n+1).$$

These relations ensure the continuity of $f(z)$ at the point $z = 0$. Therefore $f(z)$ has the form

$$f(z) = \frac{e^{-\chi(z)}}{\pi i}\int_{\partial U} \frac{\nu_1(t)t^n\,dt}{t-z}. \tag{4.43}$$

It follows immediately from the above result that $f(z) \in C_\mu(\overline{U})$.

If $n \geqslant 0$ the solution of the problem (4.41) is given by the formula

$$f(z) = \frac{z^n e^{-\chi(z)}}{2\pi i}\int_{\partial U} \nu_1(t)\frac{t+z}{t-z}\frac{dt}{t} + e^{-\chi(z)}\sum_{k=0}^{2n} C_k z^k, \tag{4.44}$$

where C_k are complex constants which satisfy the conditions

$$C_{2n-k} = -\overline{C}_k \qquad (k = 0, 1, \ldots, n).$$

It follows from (4.44) that $f(z) \in C_\mu(\overline{U})$. Thus, in view of (4.37), $w(z) \in C_\mu(\overline{U})$.

Since $w(z) \in C_\mu(\overline{U})$, $g(z) \in C^1_\mu(\overline{U})$ (Theorem 2.2); hence $\nu_0(z) \in C^1_\mu(\partial U)$ and since $q \in C^1_\mu(\partial U)$, $\chi(z) \in C^1_\mu(\overline{U})$, so that $\nu_1(z) \in C^1_\mu(\partial U)$. It follows from (4.43), (4.44) again that $f(z) \in C^1_\mu(\overline{U})$. Thus, by (4.37), $w(z) \in C^1_\mu(\overline{U})$.

By continuing a similar reasoning we conclude that $w(z) \in C_\mu^{m+1}(\overline{U})$.

5. **Generalized Schwarz reflection principle.** Let $A(z,\zeta)$, $B(z,\zeta)$, $F(z,\zeta)$ be holomorphic functions for z, ζ in a symmetric region Ω. Now we are in a position to study the relfection principle of the solutions of the following equations

$$w_{\bar{z}} = A(z,\bar{z})w(z) + B(z,\bar{z})\overline{w(z)} + F(z). \tag{5.1}$$

LEMMA 5.1. *Let Ω^+ be the part in the upper half-place of Ω, and let σ be the part of the real axis in Ω. Suppose that $w(z) = u + iv$ is a solution of the differential equation* (5.1) *in Ω^+, continuous in $\Omega^+ \cup \sigma$, satisfying*

$$\operatorname{Re}[\overline{\lambda(x)}w] \equiv \alpha u + \beta v = \rho(x), \qquad \lambda(x) = \alpha(x) + i\beta(x),$$

on σ, where $\rho(z)$, $\alpha(z)$, $\beta(z)$ are holomorphic functions in Ω, such that $\alpha(z) - i\beta(z) \neq 0$ for $z \in \Omega - \Omega^+$, $\alpha(z) + i\beta(z) = 0$ for $z \in \Omega^+ \cup \sigma$. Then $w(z)$ can be continued analytically into the domain Ω; that is, there exists a unique $w(z)$ which is a solution of (5.1) *in Ω and which agrees with the given $w(z)$ in $\Omega^+ \cup \sigma$.*

PROOF. See Yu [13].

THEOREM 5.1. *Let Ω^+ be the part in the upper half-plane of Ω, and let σ be the part of the real axis in Ω. Suppose that $w(z) = u + iv$ is a solution of* (5.1) *in Ω, $u(z)$ continuous in $\Omega^+ \cup \sigma$, satisfying*

$$u(x,0) = \rho(x), \tag{5.2}$$

where $\rho(z)$ is holomorphic in Ω. Then $w(z)$ can be continued analytically into Ω; that is, there exists a unique $w(z)$ which is a solution of (5.1) *in Ω and which agrees with the given $w(z)$ in $\Omega^+ \cup \sigma$.*

PROOF. Let D be a region with a smooth boundary ∂D such that $D \cup \partial D \subset \Omega^+ \cup \sigma$. The boundary ∂D is supposed to contain a closed segment σ_0, $\sigma_0 < \sigma$. We note that the function $\rho(x)$ is Hölder continuous on σ. Hence $u(z)$ is Hölder continuous on ∂D. By Theorem 4.1, $w(z)$ is continuous on $D \cup \sigma_0$; it therefore follows from Lemma 5.1 that $w(z)$ can be continued analytically into whole $D \cup \sigma_0 \cup \overline{D}$, where $\overline{D} = \{z \mid \bar{z} \in D\}$. But we can take D so that its boundary is as close as desired to the boundary of Ω^+. This completes the proof.

6. **Generalized M. Riesz theorem.** In this section we shall generalize the theorem of M. Riesz for conjugate functions to the solutions of the equation (1.1). The coefficients a, b, c, d, f, g of (1.1) will be assumed to be continuous in $\{|z| \leqslant 1\}$.

Unlike holomorphic functions, it is not always possible to assume $v(0) = 0$ for a solution $(u(z), v(z))$ of (1.1).

LEMMA 6.1. *The differential equations*

$$\partial v/\partial y = -bv, \qquad \partial v/\partial x = dv \tag{6.1}$$

have a nontrivial real continuous solution $v(z)$ *in* $|z| < 1$ *if and only if* b, c *satisfy the condition*

$$\int_0^y b(x,t)\,dt + \int_0^x d(t,y)\,dt = \int_0^y b(0,t)\,dt + \int_0^x d(t,0)\,dt. \tag{6.2}$$

Furthermore, let β *be an arbitrary real number; if* b, d *satisfy the condition* (6.2), *then* (6.1) *has a unique solution* $v(z)$ *such that* $v(0) = \beta$.

PROOF. The general solution of the first equation of (6.1) is given by

$$v(x,y) = k(x)\exp\left[-\int_0^y b(x,t)\,dt\right]$$

and the general solution of the second equation of (6.1) is given by

$$v(x,y) = l(y)\exp\left[\int_0^x d(t,y)\,dt\right]$$

where $k(x)$ and $l(y)$ are continuous functions in $(-1, 1)$.

It is clear that (6.1) has a solution if and only if

$$k(x)\exp\left[-\int_0^y b(x,t)\,dt\right] = l(y)\exp\left[\int_0^x d(t,y)\,dt\right]. \tag{6.3}$$

The formula (6.3) is equivalent to

$$k(x) = l(y)\exp\left[\int_0^x d(t,y)\,dt + \int_0^y b(x,t)\,dt\right].$$

Consider now $k(0) \neq 0$; then we see

$$l(y) = k(0)\exp\left[-\int_0^y b(0,t)\,dt\right]$$

and $k(0) = l(0)$.

Similarly,

$$k(x) = l(0)\exp\left[\int_0^x d(t,0)\,dt\right].$$

Hence the formula (6.2) follows from these statements.

If the functions b, d satisfy the condition (6.2), then

$$v(x,y) = \beta \exp\left[-\int_0^y b(x,t)\,dt + \int_0^x d(t,0)\,dt\right]$$
$$= \beta \exp\left[\int_0^x d(t,y)\,dt - \int_0^y b(0,t)\,dt\right]$$

is the solution of (6.1) such that $v(0) = \beta$.

LEMMA 6.2. *Let $w = u + iv$ be a solution of* (1.1) *in $|z| < 1$, $v(0) \neq 0$. Then* (1.1) *has another solution of the form $u(z) + iv_1(z)$ in $|z| < 1$, $v_1(0) = 0$ if and only if the coefficients b, d satisfy the condition* (6.2).

PROOF. Let $m(z)$ be a real solution of (6.1) such that $m(0) = -v(0)$. Then the function $u(z) + i(m(z) + v(z))$ serves the purpose.

Let $U = \{|z| < 1\}$. For any continuous function f defined on U, we put

$$M_p(f;r) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\,d\theta\right\}^{1/p}, \qquad 0 < p < \infty, \tag{6.4}$$

and

$$\|f\|_p = \sup_{0<r<1} M_p(f;r). \tag{6.5}$$

LEMMA 6.3. *If f is a holomorphic function in U, then $M_p(f;r)$ is a monotonically increasing function of r in* $[0, 1)$, $0 < p < \infty$.

PROOF. See Rudin [10, p. 330].

This suggests the following lemma.

LEMMA 6.4. *If f is a holomorphic function in U, then*

$$\|f\|_p = \lim_{r\to 1} M_p(f;r),$$

where $\|f\|_p$ is defined in (6.5).

LEMMA 6.5. *Let $G(z)$ be Hölder continuous in every $|z| \leqslant r$, $0 < r < 1$, with $\|G\|_p < \infty$, $1 < p < \infty$. If*

$$G_r(z) = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\sigma})\frac{e^{i\sigma} + z}{e^{i\sigma} - z}\,d\sigma, \tag{6.6}$$

for $|z| < r < 1$, then there exists a constant M_p independent of G and r such that

$$\|G_r(z)\|_p \leqslant M_p\|G\|_p, \qquad 1 < p < \infty. \tag{6.7}$$

Furthermore, if $G(z) \in C_\alpha(\overline{U})$, $0 < \alpha < 1$, *then* $G_r(z) \to G_1(z)$ *in* $C_\alpha(\overline{U})$.

PROOF. By the identity

$$\frac{e^{i\sigma} + z}{e^{i\sigma} - z} d\sigma = \left(-1 + \frac{2e^{i\sigma}}{e^{i\sigma} - z}\right) d\sigma = \frac{2}{i} \frac{d\zeta}{\zeta - z} - d\sigma \tag{6.8}$$

for $|\zeta| = 1$, and (6.6) then gives

$$G_r(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{2G(r\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi} \int_0^{2\pi} G(re^{i\sigma})\, d\sigma. \tag{6.9}$$

Consider

$$G_r^+(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{2G(r\zeta)}{\zeta - z} d\zeta, \qquad |z| < 1.$$

By Lemma 2.2, $G_r^+(z)$ is holomorphic in $|z| < 1$, Hölder continuous in $|z| \leqslant 1$, and

$$G_r^+(z) = 2G(rz) + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{2G(r\zeta)}{\zeta - z} d\zeta \tag{6.10}$$

for $|z| = 1$. Moreover, by Lemma 2.1 and Lemma 6.4

$$\|G_r^+(z)\|_p \leqslant C_1 \|G(r\zeta)\|_{L_p} \leqslant C_2 \|G\|_p, \tag{6.11}$$

where $\|G(r\zeta)\|_{L_p}$ denotes L_p norm for $G(r\zeta)$ on $|\zeta| = 1$, C_1 and C_2 are constants independent of r and G.

But

$$\left| \int_0^{2\pi} G(re^{i\sigma})\, d\sigma \right| \leqslant \|1\|_{L_q} \|G(re^{i\sigma})\|_{L_p} \leqslant C_3 \|G\|_p \tag{6.12}$$

where $\| \ \|_{L_p}$ denotes the L_p norm on $|z| = 1$ and $1/p + 1/q = 1$.

Applying (6.11) and (6.12) to (6.9), we get (6.7).

Now we are going to prove the second statement, since $G(z) \in C_\alpha(\overline{U})$, $C_\alpha(G(r\zeta) - G(\zeta), \partial U) \to 0$, as $r \to 1$, by Lemma 2.2 and (6.10), we have

$$C_\alpha(G_r^+(z) - G_1^+(z), \overline{U}) \to 0 \tag{6.13}$$

as $r \to 1$.

Combining (6.9) and (6.13), we have

$$C_\alpha(G_r(z) - G_1(z), \overline{U}) \to 0, \quad \text{as } r \to 1.$$

LEMMA 6.6. *Let* G *be a real valued, Hölder continuous function in* $|z| \leqslant 1$, *and*

H be a real valued Hölder continuous function in every $\{|z| \leq r\}$, $r < 1$, with $\|H\|_p < 1$, $p > 1$. If $f(z) = u(z) + iv(z)$ is holomorphic in $|z| < 1$, and satisfies

$$\operatorname{Re}(f(z)e^{-iG}) = H(z), \qquad |z| < 1, \tag{6.14}$$

then there exists a constant N_p independent of f, H such that

$$\|f\|_p \leq N_p(\|H\|_p + |f(0)|), \qquad p > 1. \tag{6.15}$$

PROOF. It is clear that the index of e^{-iG} on $|z| = r \leq 1$ is equal to 0. For $r < 1$, we have, by formula (4.4),

$$f(z) = e^{i\nu_r(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu_r(\sigma)}H(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}d\sigma + i\beta_0(r)\right], \tag{6.16}$$

for $|z| < r$, where

$$\nu_r(z) = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}d\sigma, \tag{6.17}$$

and $\beta_0(r)$ can be determined by the formula

$$\beta_0(r) = -if(0)e^{-i\nu_r(0)} + \frac{i}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im}\nu_r(\sigma)}H(re^{i\sigma})\,d\sigma. \tag{6.18}$$

It follows from (6.17) that

$$|\nu_r(0)| \leq \max_{0\leq\sigma\leq 2\pi}|G(re^{i\sigma})| \leq C_1 \tag{6.19}$$

where C_1 is a constant independent of r.

By Hölder's inequality,

$$\left|\int_0^{2\pi} e^{i\operatorname{Im}\nu_r(\sigma)}H(re^{i\sigma})\,d\sigma\right| \leq \|e^{i\omega_{1r}}\|_{L_q}\|H(rz)\|_{L_p} \leq C_2\|H\|_p \tag{6.20}$$

where $\|\ \|_{L_p}$ denotes the L_p norm on $|z| = 1$, $p > 1$, and $1/p + 1/q = 1$.

The estimates (6.19), (6.20) give at once

$$|\beta_0(r)| \leq C_3(\|H\|_p + |f(0)|) \tag{6.21}$$

where C_3 is independent of H, r and f.

Now, by Lemma 6.5, we have

$$\nu_r(rz) \to \nu_1(z), \quad \text{as } r \to 1 \tag{6.22}$$

in $C_\alpha(\bar{U})$.

The proof is completed by applying the above estimate, Lemma 6.5 and (6.22) to (6.16).

LEMMA 6.7. *If $f = 0$ outside U and $\|f\|_p < \infty$, $1 \leqslant p < \infty$, then the function*

$$h(z) = \iint_U \frac{f(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

belongs to $L_p(U)$. Furthermore, there exists a constant A_p such that

$$\|h\|_p \leqslant A_p \|f\|_p. \tag{6.23}$$

PROOF. Since $\|f\|_p < \infty$, the function

$$r \int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta \tag{6.24}$$

is integrable for r over $(0, 1)$; hence $f \in L_p(U)$.

If $p = 1$, we see

$$\begin{aligned} |h(z)| &\leqslant \iint_U \frac{|f(\zeta)|}{|\zeta - z|}\, d\xi\, d\eta = \int_0^1 \int_{-\pi}^{\pi} \frac{|f(\rho e^{i\sigma})|\rho}{|\rho e^{i\sigma} - z|}\, d\rho\, d\sigma \\ &= \int_{-\pi}^{\pi} \int_0^1 \frac{|f(\rho e^{i\sigma})|\rho}{|\rho - ze^{-i\sigma}|}\, d\rho\, d\sigma. \end{aligned} \tag{6.25}$$

Let $a + ib = ze^{-i\sigma} = re^{i(\theta - \sigma)}$; we have

$$\begin{aligned} &\int_0^1 \int_{-\pi}^{\pi} \frac{\rho\, d\rho\, d\sigma}{|\rho - ze^{-i\sigma}|} \\ &\qquad = \int_0^1 \int_{-\pi}^{\pi} \frac{\rho\, d\rho\, d\sigma}{\sqrt{(\rho - r\cos(\theta - \sigma))^2 + r^2 \sin^2(\theta - \sigma)}} \\ &\qquad = \int_0^1 \int_{-\pi}^{\pi} \frac{\rho\, d\rho\, d\sigma}{\sqrt{(\rho - r\cos(\sigma - \theta))^2 + r^2 \sin^2(\sigma - \theta)}} < C_1 \end{aligned} \tag{6.26}$$

so that

$$\begin{aligned} \int_{-\pi}^{\pi} |h(re^{i\theta})|\, d\theta &\leqslant \int_{-\pi}^{\pi} |f(\rho e^{i\sigma})|\, d\sigma \int_0^1 \int_{-\pi}^{\pi} \frac{\rho\, d\rho\, d\theta}{|\rho - ze^{-i\sigma}|} \\ &\leqslant C_1 \|f\|_1. \end{aligned} \tag{6.27}$$

If $p > 1$, by the Hölder inequaltiy, for $1/p + 1/q = 1$,

$$|h(re^{i\theta})| \leqslant \int_0^1 \int_{-\pi}^{\pi} \frac{|f(\rho e^{i\sigma})|\rho}{|\rho - ze^{-i\sigma}|} d\rho\, d\sigma$$

(6.28)

$$\leqslant \left(\int_0^1 \int_{-\pi}^{\pi} \frac{|f(\rho e^{i\sigma})|^p \rho}{|\rho - ze^{-i\sigma}|} d\rho\, d\sigma\right)^{1/p} \left(\int_0^1 \int_{-\pi}^{\pi} \frac{\rho}{|\rho - ze^{-i\sigma}|} d\rho\, d\sigma\right)^{1/q}.$$

By (6.27), we see

$$\int_{-\pi}^{\pi} |h(re^{i\theta})|^p d\theta \leqslant C_2 \|f\|_p^p. \tag{6.29}$$

We now proceed to the generalized theorem of M. Riesz. In the case that the coefficients b, d of (1.1) satisfy the condition (6.2) we have the following generalization.

THEOREM 6.1. *Suppose that the coefficients b, d of* (1.1) *satisfy the condition* (6.2). *Then to each p such that $1 < p < \infty$ there correspond two constants A_p and B_p, such that the inequality*

$$\|v(z)\|_p \leqslant A_p \|u(z)\|_p + B_p \tag{6.30}$$

holds for every solution $w(z) = u(z) + iv(z)$ of (1.1) *on U, where $v(0) = 0$. Moreover, $B_p = 0$ if $f \equiv g \equiv 0$.*

REMARK. By Lemma 6.2, it is always possible to assume $v(0) = 0$ in the statement of the above theorem.

PROOF OF THEOREM 6.1. *First Method.* Since $\|u\|_p < \infty$, $1 < p < \infty$, $u \in L_p(U)$. By Lemma 3.1 (the Basic Lemma),

$$w(z) = \frac{f(z)}{\mu(z)} - \frac{1}{\pi\mu} \iint_U \frac{2B\mu u + \mu F}{\zeta - z} d\xi\, d\eta \tag{6.31}$$

for $z \in U$, where $f(z)$ is a holomorphic function in U. Therefore

$$u(z) = \operatorname{Re}\left(\frac{f(z)}{\mu(z)}\right) - \operatorname{Re} \frac{1}{\pi\mu(z)} \iint_U \frac{2B\mu u + \mu F}{\zeta - z} \tag{6.32}$$

for $z \in U$. Setting

$$e^{K-iG} = \frac{1}{\mu(z)} \tag{6.33}$$

and

$$\beta(z) = e^{-K(z)}\left\{u(z) + \operatorname{Re} \frac{1}{\pi\mu(z)} \iint_U \frac{2B\mu u + \mu F}{\zeta - z} d\xi\, d\eta\right\}, \tag{6.34}$$

we have

$$\text{(6.35)} \qquad \operatorname{Re}\{f(z)e^{-iG(z)}\} = \beta(z)$$

for $z \in U$.

Since u is a real part of $w(z)$, by Theorem 2.6, $\beta(z) \in C_\alpha(\overline{U}_r)$, $U_r = \{|z| < r < 1\}$. Therefore, the condition (6.35) fulfills the requirement of Lemma 6.6; hence

$$\text{(6.36)} \qquad \|f\|_p \leqslant N_p(\|\beta\|_p + |f(0)|)$$

and by Lemma 6.7, we have

$$\text{(6.37)} \qquad \|\beta\|_p \leqslant C_1(\|u\|_p + \|F\|_p).$$

Because $v(0) = 0$,

$$\text{(6.38)} \qquad f(0) = \mu(0)u(0) + \frac{1}{\pi}\iint_U \frac{2B\mu u + \mu F}{\zeta}\,d\xi\,d\eta$$

and hence (Lemma 6.7),

$$\text{(6.39)} \qquad |f(0)| \leqslant C_2(\|u\|_p + \|F\|_p).$$

Applying the estimates (6.38) and (6.39) to (6.37), we see

$$\text{(6.40)} \qquad \|f\|_p \leqslant C_3(\|u\|_p + \|F\|_p).$$

Still using Lemma 6.7, we see

$$\text{(6.41)} \qquad \left\| \frac{1}{\pi\mu}\iint_U \frac{2B\mu u + \mu F}{\zeta - z}\,d\xi\,d\eta \right\|_p \leqslant C_4(\|u\|_p + \|F\|_p)$$

and therefore the desired inequality (6.30) follows from (6.31), (6.40) and (6.41).

Second Method. The solution $w(z)$ of (1.1) on U is representable in the form $w = w_0 + w_1$ where w_0 is a solution of the homogeneous equation $\partial_{\bar z} w = Aw + B\overline{w}$ and consequently it is given by the relation (Lemma 3.2)

$$\text{(6.42)} \qquad w_0(z) = f(z)e^{\omega(z)}, \qquad \omega(z) = \frac{1}{\pi}\iint_G \left(A + B\frac{\overline{w}_0}{w_0}\right)\frac{d\xi\,d\eta}{\zeta - z},$$

w_1 is a particular solution of the equation (2.1), which may be taken in the form (Lemma 3.3)

$$\text{(6.43)} \quad w_1(z) = -\frac{1}{\pi}\iint_U \Omega_1(z,\zeta)F(\zeta)\,d\xi\,d\eta - \frac{1}{\pi}\iint_U \Omega_2(z,\zeta)\overline{F(\zeta)}\,d\xi\,d\eta.$$

The function ω and $w_1 \in C_\alpha(\bar{U})$, $0 < \alpha < 1$, and the function $f(z)$ is holomorphic in U and satisfies the condition

$$\text{Re}[e^{\omega(z)} f(z)] = u(z) - \text{Re}\, w_1(z), \qquad z \in U. \tag{6.44}$$

Setting

$$\omega(z) = G_1(z) - iG(z), \tag{6.45}$$

we obtain

$$\text{Re}[e^{-iG(z)} f(z)] = \beta(z) \qquad (z \in U) \tag{6.46}$$

where

$$\beta(z) = u(z)e^{-G_1(z)} - \text{Re}\, w_1(z)e^{-G_1(z)}. \tag{6.47}$$

It follows immediately from the above result (Lemma 6.6) that $\|f\|_p < \infty$, $p > 1$. Moreover,

$$\|f\|_p \leqslant N_p(\|\beta\|_p + |f(0)|). \tag{6.48}$$

Since $v(0) = 0$,

$$f(0) = e^{-\omega(0)}(u(0) - w_1(0)). \tag{6.49}$$

Thus

$$|f(0)| \leqslant C_1(\|u\|_p + \|w_1\|_p). \tag{6.50}$$

Evidently,

$$\|\beta(z)\|_p \leqslant C_2(\|u\|_p + \|w_1\|_p). \tag{6.51}$$

Hence

$$\|f\|_p \leqslant C_3(\|u\|_p + \|w_1\|_p). \tag{6.52}$$

Introducing the formula (6.52) into (6.42) we obtain

$$\|w\|_p \leqslant A_p(\|u\|_p + \|w_1\|_p). \tag{6.53}$$

This completes the proof.

If the coefficients b, d of (1.1) do not satisfy the condition (6.2), we cannot always assume $v(0) = 0$ for a solution $(u(z), v(z))$ of (1.1). But surprisingly we still can extend the theorem of M. Riesz to the solutions of (1.1).

THEOREM 6.2. *Suppose that the coefficients b, d of* (1.1) *do not satisfy the condition* (6.2). *Then to each p such that* $1 < p < \infty$ *there correspond two constants* A_p *and* B_p *such that the inequality*

$$\|v\|_p \leqslant A_p\|u\|_p + B_p \tag{6.54}$$

holds for every solution $w(z) = u(z) + iv(z)$ *of* (1.1) *on U. Moreover,* $B_p = 0$ *iff* $f \equiv g \equiv 0$.

PROOF. *First Method.* As we did in the proof of Theorem 6.1, we again have formulas (6.31), (6.32), (6.33), (6.34) and (6.35).

First, we show that $f(z)$ in (6.32) is the only holomorphic function satisfying the above statement. Suppose that $f_1(z)$ is a holomorphic function in U such that $\operatorname{Im}(f_1(z)/\mu(z)) \equiv 0$. Because $f_1(z)/\mu(z)$ is a real solution of the equation

$$\frac{\partial v}{\partial \bar{z}} = (B - A)v, \tag{6.56}$$

which is the complex form of the equations (6.1) for real solutions; from Lemma 6.1, $f_1(z)/\mu(0) \equiv 0$. Then we must have $f_1(z) \equiv 0$.

Next, we want to show that there exists a constant N_p independent of u such that

$$\|f\|_p \leqslant N_p N_p(\|u\|_p + \|F\|_p). \tag{6.56}$$

By formula (4.4), we have

$$\begin{aligned} f(z) &= e^{iv_r(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im} v_r(\sigma)}\beta(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}d\sigma + i\beta_0(r)\right] \\ &= i\beta_0(r)e^{iv_r(z)} + g_r(z) \end{aligned} \tag{6.57}$$

for $|z| < r < 1$, where

$$v_r(z) = \frac{1}{2\pi}\int_0^{2} G(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}d\sigma. \tag{6.58}$$

Therefore, from (6.36)

$$\operatorname{Re}\{i\beta_0(r)e^{iv_r(z)-iG(z)}\} = \beta(z) - \operatorname{Re}\{g_r(z)e^{-iG(z)}\} \tag{6.59}$$

for $|z| < r < 1$.

Since $G(z)$ is Hölder continuous in $|z| \leqslant 1$, by Lemma 6.5

$$v_r(rz) \to v_1(z), \quad \text{as } r \to 1, \tag{6.60}$$

in $C_\alpha(\overline{U})$. Then

$$\|\operatorname{Re} ie^{i\nu_r(rz)-iG(rz)} - \operatorname{Re} ie^{i\nu_r(z)-iG(z)}\|_p \to 0 \tag{6.61}$$

as $r \to 1$.

We can show the following inequality

$$\|\operatorname{Re} ie^{i\nu_1(z)-iG(z)}\|_p > 0 \tag{6.62}$$

by contradiction. If we assume that (6.62) is false, then $\operatorname{Re}(ie^{i\nu_1(z)-iG}) \equiv 0$ and the function $e^{i\nu_1(z)+F-iG}$ is a real solution of (6.55), then from Lemma 6.1, we have $e^{i\nu_1(z)+F-iG} \equiv 0$, and therefore we must have $e^{i\nu_1(z)} \equiv 0$.

From (6.61) and (6.62) it follows that there exist two constants $d_1, r_1 > 0$ such that

$$\|\operatorname{Re} ie^{i\nu_r(rz)-iG(rz)}\|_p > d_1 \tag{6.63}$$

for $r_1 \leqslant r \leqslant 1$.

On the other hand, by Lemma 2.1 and Lemma 2.2, we see

$$\|\beta(rz) - \operatorname{Re} g_r(rz)e^{-iG(rz)}\|_p < d_2\|\beta\|_p, \tag{6.64}$$

where d_2 is a constant independent of r.

Therefore from (6.59), (6.62), (6.63) and (6.64) we have

$$|\beta_0(r)| \leqslant \frac{d_2}{d_1}\|\beta\|_p \leqslant d_2(\|u\|_p + \|F\|_p). \tag{6.65}$$

Applying (6.60), (6.65) and Lemma 6.5 to (6.57), we get the estimate

$$\|f\|_p \leqslant N_p(\|u\|_p + \|F\|_p). \tag{6.66}$$

The desired inequality (6.54) follows from (6.31), (6.41) and (6.66).

Second Method. As we did in the second method of the proof of Theorem 6.1, we again have formulas (6.42), (6.43), (6.44), (6.45), (6.46) and (6.47).

Also as we did in the first method of the proof of this theorem, we get the estimate

$$\|f\|_p \leqslant N_p(\|u\|_p + \|w_1(z)\|). \tag{6.67}$$

The desired inequality (6.30) follows from (6.42) and (6.67).

7. **Generalized Kolmogorov's theorem.** In this section we shall generalize the Kolmogorov's theorem for conjugate functions to the solutions of the equation (1.1). The coefficients a, b, c, d, f, g of (1.1) will be assumed to be continuous in $\overline{U}$.

It is convenient to retain the norm notation (6.5) for $p < 1$, when $\| \|_p$ is not a genuine norm.

LEMMA 7.1. *Let*

$$G_r(z) = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma. \tag{7.1}$$

If $G(z)$ is Hölder continuous for every $|z| \leqslant r$, $r < 1$ and $\|G\|_1 < \infty$, then there exists a constant M_p independent of G and r such that

$$\|G_r(z)\|_p \leqslant M_p\|G\|_1, \qquad 0 < p < 1. \tag{7.2}$$

PROOF. The method in this proof is similar to that in the proof of Lemma 6.5. Using the same notation, we find by Hölder inequality

$$\left|\int_0^{2\pi}|G(re^{i\sigma})|^p\,d\sigma\right| \leqslant \left(\int_0^{2\pi}|G(re^{i\sigma})|^{1/p}\,d\sigma\right)^p\left(\int_0^{2\pi}d\sigma\right)^q \leqslant d_1\|G\|_1 \tag{7.3}$$

where $p + q = 1$.

By Lemma 2.4, Lemma 6.4 and (7.3), we see

$$\|G_r^+(z)\|_p \leqslant d_2\|G(rz)\|_{L_1} \leqslant d_3\|G\|_1 \tag{7.4}$$

where $\|G(rz)\|_{L_1}$ denotes L_1 norm for $G(rz)$ on $|z| = 1$, d_2 and d_3 are constants independent of r and G.

Applying (7.3) and (7.4) to (6.9), we have inequality (7.2).

LEMMA 7.2. *Let F, G be real valued, Hölder continuous functions in $|z| \leqslant 1$, and let H be a real valued Hölder continuous function in every $\{|z| < r\}$, $r < 1$, with $\|H\|_1 < \infty$. If $f(z) = u(z) + iv(z)$ is holomorphic in $|z| < 1$, and satisfies*

$$\operatorname{Re}(f(z)e^{F-iG}) = H(z) \tag{7.5}$$

for $|z| < 1$, then there exists a constant N_p independent of f, H such that

$$\|f\| \leqslant N_p(\|H\|_1 + |f(0)|), \qquad p < 1. \tag{7.6}$$

PROOF. The method in this proof is similar to that in the proof of Lemma 6.6. Using the same notation, we again have estimate (6.19) and

$$\left|\int_0^{2\pi} e^{i\operatorname{Im}\gamma_r(\sigma)}H(re^{i\sigma})\,d\sigma\right| \leqslant \|H(rz)\|_{L_1} \leqslant d_1\|H\|_1 \tag{7.7}$$

where $\| \|_{L_1}$ denotes the L_1 norm on $|z| = 1$, and d_2 is a constant independent of H.

The estimate (6.19), (7.7) give at once

$$|\beta_0(r)| \leqslant d_2(\|H\|_1 + |f(0)|) \tag{7.8}$$

where d_2 is independent of H and r.

Now, by Lemma 7.1, we have

$$\left\| \frac{1}{2\pi}\int_0^{2\pi} e^{i\operatorname{Im} v_r(\sigma)} H(re^{i\sigma}) \frac{re^{i\sigma}+z}{re^{i\sigma}-z}\, d\sigma \right\|_p \leqslant d_3\|H\|_1. \tag{7.9}$$

The proof is therefore completed by applying (7.8), (7.9) and (6.22) to (6.16).

LEMMA 7.3. *Let*

$$h_r(z) = \iint_{U_r} \frac{f(\zeta)}{\zeta - z}\, d\xi\, d\eta. \tag{7.10}$$

If $\|f\|_1 < \infty$, *and* $f = 0$ *outside* U, *then there exists a constant* C_p *such that*

$$\|h_r(rz)\|_p \leqslant C_p\|f\|_1 \tag{7.11}$$

where $U_r = \{|z| < r \leqslant 1\}$, $0 < p \leqslant 1$.

PROOF.

$$\begin{aligned} h_r(rz) &= \iint_{U_r} \frac{f(\zeta)}{\zeta - rz}\, d\xi\, d\eta \\ &= \iint_U \frac{f(r\zeta')}{r\zeta' - rz} r^2\, d\xi'\, d\eta' = \iint_U \frac{f(r\zeta')r}{\zeta' - z}\, d\xi\, d\eta. \end{aligned} \tag{7.12}$$

Applying Lemma 6.7 for $h_r(rz)$, we see

$$\|h_r(rz)\|_1 \leqslant A_1\|rf(r\zeta')\|_1 \leqslant A_1\|f\|_1. \tag{7.13}$$

Using Hölder's inequality, we obtain

$$\int_{-\pi}^{\pi} |h_r(re^{i\theta})|^p\, d\theta \leqslant \left(\int_{-\pi}^{\pi} d\theta\right)^{1-p} \left(\int_{-\pi}^{\pi} |h_r(re^{i\theta})|\, d\theta\right)^p \tag{7.14}$$

for $0 < p < 1$. Thus

$$\|h_r(rz)\|_p \leqslant B_p\|h_r(rz)\|_1. \tag{7.15}$$

Collecting inequalities (7.13), (7.15), we get (7.11).

LEMMA 7.4. *For arbitrary positive numbers a and b*

$$(a+b)^p \leqslant \begin{cases} a^p + b^p, & 0 < p < 1, \\ 2^{p-1}(a^p + b^p), & p > 1. \end{cases} \tag{7.16}$$

PROOF. It is well known.

If the coefficients b, d of (1.1) satisfy the condition (6.2), then we have the following generalized Kolmogorov's theorem.

THEOREM 7.1. *Suppose that the coefficients b, d of* (1.1) *satisfy the condition* (6.2). *Then to each p such that* $0 < p < 1$ *there correspond two constants A_p and B_p such that the inequality*

$$\|v(z)\|_p \leqslant A_p \|u(z)\|_1 + B_p \tag{7.17}$$

holds for every solution $w(z) = u(z) + iv(z)$ of (1.1) *on U, where $v(0) = 0$, and $\| \ \|_p$ is as in* (6.5). *Furthermore, $B_p = 0$ if $f \equiv g \equiv 0$.*

PROOF. *First Method.* By Lemma 3.1,

$$w(z) = \frac{f_r(z)}{\mu(z)} - \frac{1}{\pi\mu} \iint_{U_r} \frac{2B\mu u + \mu F}{\zeta - z} d\xi \, d\eta, \tag{7.18}$$

for $z \in U_r$, where $U_r = \{|z| < r < 1\}$, and $f_r(z)$ is a holomorphic function in U_r.

Setting

$$e^{K - iG} = \frac{1}{\mu(z)} \tag{7.19}$$

and

$$\rho_r(z) = e^{-K(z)} \left\{ u(z) + \operatorname{Re} \frac{1}{\pi\mu(z)} \iint_{U_r} \frac{2B\mu u + \mu F}{\zeta - z} d\xi \, d\eta \right\}, \tag{7.20}$$

from (7.18) we have

$$\operatorname{Re}\{f_r(z) e^{-iG(z)}\} = \rho_r(z), \tag{7.21}$$

for $z \in U_r$.

Since u is a real part of $w(z)$, by Theorem 2.6, $\rho_r(z) \in C_\alpha(\overline{U}_r)$, $0 < \alpha < 1$. Therefore, according to Lemma 7.2

$$\|f_r(rz)\|_p \leqslant d_1(\|\rho_r(rz)\|_1 + |f(0)|). \tag{7.22}$$

According to Lemma 7.3

$$\|\rho_r(rz)\|_1 \leqslant d_2(\|u\|_1 + \|F\|_1), \tag{7.23}$$

still using Lemma 7.3, we see

$$(7.24) \qquad \left\| \frac{1}{\pi\mu(z)} \int_{U_r} \frac{2B\mu u + \mu F}{\zeta - rz} d\xi\, d\eta \right\|_p \leqslant d_3(\|u\|_1 + \|F\|_1),$$

where the constant d_3 is independent of r, u, and F.

Applying (7.22), (7.23) and (7.24) to (7.18), it follows that

$$(7.25) \qquad \|w(rz)\|_1 \leqslant d_4(\|u\|_1 + \|F\|_1).$$

Hence the desired inequality (7.17) follows.

Second Method. This method is similar to the second method used for the proof of Theorem 6.1.

If the coefficients b, d of (1.1) do not satisfy the condition (6.2), we cannot always assume $v(0) = 0$ for a solution $(u(z), v(z))$ of (1.1) in U. Then the generalized Kolmogorov's theorem has the following version.

THEOREM 7.2. *Suppose that the coefficients b, d of* (1.1) *do not satisfy the condition* (6.2). *Then to each p such that $0 < p < 1$ there correspond two constants A_p and B_p such that*

$$(7.26) \qquad \|v(z)\|_p \leqslant A_p \|u(z)\|_1 + B_p$$

holds for every solution $w(z) = u(z) + iv(z)$ of (1.1) *on U, and $\| \; \|_p$ is defined as in* (6.5). *Moreover, $B_p = 0$ if $f \equiv g \equiv 0$.*

PROOF. *First Method.* We first proceed in the same way as we did in the proof of Theorem 7.1 to obtain (7.18), (7.19), (7.20) and (7.21).

According to Theorem 2.6, $\rho_r(z) \in C_\alpha(\overline{U}_r)$, $0 < \alpha < 1$, $r < 1$.

By formula (4.4), we have

$$(7.27) \qquad \begin{aligned} f_r(z) &= e^{i\nu_r(z)}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{i\,\mathrm{Im}\,\nu_r(\sigma)}\rho_r(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}\,d\sigma + i\beta_0(r)\right] \\ &= i\beta_0(r)e^{i\nu_r(z)} + g_r(z) \end{aligned}$$

for $|z| < r < 1$, where

$$(7.28) \qquad \nu_r(z) = \frac{1}{2\pi}\int_0^{2\pi} G(re^{i\sigma})\frac{re^{i\sigma}+z}{re^{i\sigma}-z}\,d\sigma.$$

Therefore, from (7.20) we deduce that

$$(7.29) \qquad \mathrm{Re}\{i\beta_0(r)e^{i\nu_r(z) - iG(z)}\} = \rho_r(z) - \mathrm{Re}\{g_r(z)e^{-iG(z)}\}.$$

Since $G(z)$ is Hölder continuous in $|z| \leqslant 1$, by Lemma 7.1,

$$C_\alpha(\nu_r(rz) - \nu_1(z), \overline{U}) \to 0, \quad \text{as } r \to 1.$$

Therefore,

$$\text{Im } e^{i\nu_r(rz) - iG(rz)} - \text{Im } e^{i\nu_1(z) - iG(z)} \to 0 \tag{7.30}$$

uniformly as $r \to 1$ for $z \in \overline{U}$. Thus

$$\|\text{Im } e^{i\nu_r(rz) - iG(rz)} - \text{Im } e^{i\nu_1(z) - iG(z)}\|_p \to 0 \tag{7.31}$$

as $r \to 1$.

Since $e^{i\nu_1(z)}/\mu(z)$ is a nontrivial solution of

$$\frac{\partial v}{\partial \overline{z}} = (B - A)v, \tag{7.32}$$

it follows that $\|\text{Im } e^{i\nu_1(z) - iG(z)}\|_p > 0$; otherwise $e^{i\nu_1(z)}/\mu(z)$ is a nontrivial real solution of (6.1); this contradicts Lemma 6.1.

Therefore, by Lemma 7.4, there exist two positive numbers C_1, r_1 such that

$$\|\text{Im } e^{i\nu_r(rz) - iG(rz)}\|_p > C_1 > 0 \tag{7.33}$$

for $r_1 \leqslant r < 1$.

On the other hand, by Lemma 7.3 and Lemma 7.4, we get

$$\|\rho_r(rz)\|_1 \leqslant C_2(\|u\|_1 + \|F\|_1) \tag{7.34}$$

and

$$\|\rho_r(rz)\|_p \leqslant C_3(\|u\|_1 + \|F\|_1). \tag{7.35}$$

By Lemma 2.4, (7.19) and (7.34), we see

$$\left\| g_r(rz)\frac{1}{\mu(rz)} \right\|_p \leqslant C_4\|\rho_r(rz)\|_1 \leqslant C_5(\|u\|_1 + \|F\|_1). \tag{7.36}$$

Therefore, we have

$$\left\| \rho_r(rz) - \text{Re } g_r(rz)\frac{1}{\mu(rz)} \right\|_p \leqslant C_6(\|u\|_1 + \|F\|_1). \tag{7.37}$$

Thus, from (7.21), we deduce that

$$|\beta_0(r)| \leqslant \frac{C_6}{C_1}(\|u\|_1 + \|F\|_1). \tag{7.38}$$

Applying (7.38), Lemma 2.4 and (7.34) to (7.27), we get the estimate

$$\|f_r(rz)\|_p \leqslant C_7(\|u\|_1 + \|F\|_1), \tag{7.39}$$

where C_7 is independent of r, u, and F. Hence the desired inequality (7.25) follows from (7.18), (7.24) and (7.39).

Second Method. This method is similar to the second method used for the proof of Theorem 6.2.

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